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The inverse scattering problem for a reflectional system

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Abstract. In this paper, the general solution to the inverse scattering problem for a reflectional system is formulated. Two families of eigenfunctions, corresponding to the continuous and discrete spectra, are introduced to transform the Gel'fand–Levitan integral equation to a system of $N+1$ equations. Making use of the available soliton solution, it is shown that the inverse scattering problem can be reduced to one of solving the eigenfunction for a continuous spectrum only. In addition, the properties of these eigenfunctions are also investigated.

1. Introduction

In recent years there has been considerable interest in certain classes of nonlinear partial differential equations which describe a wide variety of physical models (Ablowitz *et al* 1973, Hirota 1973, Kingston and Rogers 1975, McLaughlin 1975). Through transformation these equations can be associated with the linear Schrödinger operator in an inverse manner. The inverse scattering method may thus be employed to solve these initial-value problems.

During the past two decades, the inverse problem of the Schrödinger operator has been extensively investigated by numerous authors. In particular, Gel'fand and Levitan (1955) reduced this problem to a linear integral equation. Kay and Moses (1956a) treated the reflectionless system, and the N-soliton solution was thus obtained (Hirota 1971). Ablowitz and Newell (1973) investigated the asymptotic behaviour of the solution for a system with continuous spectrum.

In the present paper, the general solution to the inverse scattering problem which covers the entire spectrum is formulated. Two families of eigenfunctions, corresponding to the continuous and discrete spectra, are introduced to transform the Gel'fand–Levitan integral equation to a system of $N+1$ equations. Making use of the available soliton solution, these $N+1$ equations can be solved to yield one linear integral equation for the eigenfunction pertaining to continuous spectrum only. This equation may be employed to investigate the interaction of oscillatory waves and solitons.

2. Preliminary remarks

In this section, we shall briefly outline some of the previous results, which will be referred to in this paper. Let us consider the inverse scattering problem

$$\psi_{xx}+(V+\lambda^2)\psi=0 \tag{2.1}$$

subject to the condition that V vanishes at infinity. This implies the asymptotic conditions

$$\psi(x, \lambda) = A(\lambda, t)\,e^{i\lambda x} + B(\lambda, t)\,e^{-i\lambda x}, \tag{2.2}$$

where the eigenvalue λ may be either real or imaginary. Here $V(x, t)$ also depends parametrically on t as governed by a nonlinear partial differential equation

$$L_{t,x}[V] = 0, \tag{2.3}$$

where L is a nonlinear partial differential operator which can be associated with the linear Schrödinger operator through a certain transformation. The coefficients $A(\lambda, t)$ and $B(\lambda, t)$ may be determined from equation (2.3).

Suppose the function $F(x, y \leq x)$ exists, having continuous partial derivatives of first and second orders, such that (Agranovich and Marchenko 1963)

$$\psi(x, \lambda; t) = A(\lambda, t)\left(e^{i\lambda x} - \int_{-\infty}^{x} F(x, y; t)\,e^{-i\lambda y}\,dy\right), \tag{2.4}$$

where $A(\lambda, t)$ is the normalization coefficient. Then, the solution for the inverse scattering problem can be related to F by

$$V(x, t) = 2\frac{d}{dx}F(x, x; t). \tag{2.5}$$

Gel'fand and Levitan (1955) have reduced this problem to a linear integral equation

$$F(x, y) + \int_{-\infty}^{x} F(x, z)R(y+z)\,dz = R(x+y), \tag{2.6}$$

where R is a known function determined by the asymptotic behaviour of ψ. The function R can be represented in terms of the reflection coefficient (Kay and Moses 1956b):

$$R(x+y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} b(k, t)\,e^{-ik(x+y)}\,dk + \sum_{n=1}^{N} c_n^2(t)\,e^{\kappa_n(x+y)}. \tag{2.7}$$

Here the reflection coefficient may be defined as

$$b(\lambda, t) = B(\lambda, t)/A(\lambda, t), \tag{2.8}$$

with residues at the simple poles $\lambda = i\kappa_n$ denoted by $ic_n^2(t)$.

3. General solution

For a reflectional system, we assume that the solution of the Gel'fand–Levitan integral equation, for $y \leq x$, has the form

$$F(x, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(k, x)\,e^{-iky}\,dk + \sum_{n=1}^{N} G_n(x)\,e^{\kappa_n y} \tag{3.1}$$

where the eigenfunctions $H(k, x)$ are complex, while $G_n(x)$ are real, and $\kappa_n > 0$.

Substituting expression (3.1) into equations (2.4) and (2.5), we obtain the expressions for ϕ and V:

$$\phi(x,\lambda)=A(\lambda,t)\,\mathrm{e}^{\mathrm{i}\lambda x}\left(1-\frac{1}{2\pi}\lim_{\epsilon\to 0}\int_{-\infty}^{\infty}\frac{H(k,x)\,\mathrm{e}^{-ikx}}{\epsilon+\mathrm{i}(\lambda-k)}\mathrm{d}k-\sum_{n=1}^{N}\frac{G_n(x)}{\kappa_n+\mathrm{i}\lambda}\mathrm{e}^{\kappa_n x}\right), \tag{3.2}$$

$$V(x)=2\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}H(k,x)\,\mathrm{e}^{-ikx}\,\mathrm{d}k+\sum_{n=1}^{N}G_n(x)\,\mathrm{e}^{\kappa_n x}\right). \tag{3.3}$$

Then, substituting expressions (3.2) and (3.3) into equation (2.1), we have

$$\frac{1}{2\pi}\lim_{\epsilon\to 0}\int_{-\infty}^{\infty}\mathrm{d}k\frac{\mathrm{e}^{-ikx}}{\epsilon+\mathrm{i}(\lambda-k)}\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}+V+k^2\right)H(k,x)$$
$$+\sum_{n=1}^{N}\frac{\mathrm{e}^{\kappa_n x}}{\kappa_n+\mathrm{i}\lambda}\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}+V-\kappa_n^2\right)G_n(x)=0.$$

Since κ_n are arbitrary, one may expect that

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}+V+k^2\right)H(k,x)=0, \tag{3.4}$$

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}+V-\kappa_n^2\right)G_n(x)=0. \tag{3.5}$$

In order to solve the eigenfunctions $H(k,x)$ and $G_n(x)$, we shall substitute expressions (3.1) and (2.7) into equation (2.6). Let us first evaluate the integral

$$\int_{-\infty}^{x}F(x,z)R(y+z)\,\mathrm{d}z. \tag{3.6}$$

Since the step function, expressed as

$$\theta(x)=\frac{1}{2\pi}\lim_{\epsilon\to 0}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}\omega x}}{\epsilon+\mathrm{i}\omega}\mathrm{d}\omega,$$

may be introduced to transform an indefinite integral into a definite one, we have

$$\int_{-\infty}^{x}\mathrm{e}^{-\mathrm{i}(k+k')z}\,\mathrm{d}z=\int_{-\infty}^{\infty}\theta(x-z)\,\mathrm{e}^{-\mathrm{i}(k+k')z}\,\mathrm{d}z=\lim_{\epsilon\to 0}\frac{\mathrm{e}^{-\mathrm{i}(k+k')x}}{\epsilon-\mathrm{i}(k+k')}.$$

Thus, the integral (3.6) can be evaluated and written as

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}k\,b(k,t)\,\mathrm{e}^{-ik(x+y)}\left(\frac{1}{2\pi}\lim_{\epsilon\to 0}\int_{-\infty}^{\infty}\mathrm{d}k'\frac{H(k',x)\,\mathrm{e}^{-ik'x}}{\epsilon-\mathrm{i}(k'+k)}+\sum_{n=1}^{N}\frac{G_n(x)}{\kappa_n-ik}\mathrm{e}^{\kappa_n x}\right)$$
$$+\sum_{n=1}^{N}c_n^2(t)\,\mathrm{e}^{\kappa_n(x+y)}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}k\frac{H(k,x)}{\kappa_n-ik}\mathrm{e}^{-ikx}+\sum_{m=1}^{N}\frac{G_m(x)}{\kappa_m+\kappa_n}\mathrm{e}^{\kappa_m x}\right).$$

Equation (2.6) then becomes

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{-iky}\Big[H(k,x)+b(k,t)\,e^{-ikx}\Big(\frac{1}{2\pi}\lim_{\epsilon\to 0}\int_{-\infty}^{\infty} dk'\frac{H(k',x)\,e^{-ik'x}}{\epsilon-i(k'+k)}$$

$$+\sum_{n=1}^{N}\frac{G_n(x)}{\kappa_n-ik}e^{\kappa_n x}-1\Big)\Big]$$

$$+\sum_{n=1}^{N} e^{\kappa_n y}\Big[G_n(x)+c_n^2(t)\,e^{\kappa_n x}\Big(\frac{1}{2\pi}\int_{-\infty}^{\infty} dk\frac{H(k,x)}{\kappa_n-ik}e^{-ikx}$$

$$+\sum_{m=1}^{N}\frac{G_m(x)}{\kappa_m+\kappa_n}e^{\kappa_m x}-1\Big)\Big]=0.$$

This gives

$$H(k,x)=b(k,t)\,e^{-ikx}\Big(1-\frac{1}{2\pi}\lim_{\epsilon\to 0}\int_{-\infty}^{\infty}\frac{H(k',x)\,e^{-ik'x}}{\epsilon-i(k'+k)}dk'-\sum_{n=1}^{N}\frac{G_n(x)}{\kappa_n-ik}e^{\kappa_n x}\Big) \tag{3.7}$$

and

$$G_n(x)=c_n^2(t)\,e^{\kappa_n x}\Big(1-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{H(k,x)}{\kappa_n-ik}e^{-ikx}dk-\sum_{m=1}^{N}\frac{G_m(x)}{\kappa_m+\kappa_n}e^{\kappa_m x}\Big). \tag{3.8}$$

Due to the composite expression of $R(x+y)$, given by expression (2.7), the exact solution of the Gel'fand–Levitan integral equation for a non-zero reflection coefficient is hardly attainable. This integral equation has been transformed to a system of $N+1$ equations so that the problem may be tackled as shown in § 4.

Theorem 1. Equation (3.4) is satisfied by $H(k, x)$ as defined in equation (3.7), and equation (3.5) is satisfied by $G_n(x)$ as defined in equation (3.8), while $V(x)$ is given by expression (3.3).

Proof. By expression (3.2), equations (3.7) and (3.8) may be expressed simply as

$$H(k,x)=\frac{b(k,t)}{A^*(k,t)}\psi^*(x,k) \tag{3.9}$$

and

$$G_n(x)=c_n(t)\psi_n(x), \tag{3.10}$$

where the asterisk denotes the complex conjugate, ψ_n denotes $\psi(x,-i\kappa_n)$, and $A(-i\kappa_n,t)=c_n(t)$. Equation (2.1) shows that $\psi^*(x,k)$ satisfies the differential equation

$$\psi^*_{xx}+(V+k^2)\psi^*=0 \tag{3.11}$$

and $\psi_n(x)$ satisfies

$$(\psi_n)_{xx}+(V-\kappa_n^2)\psi_n=0. \tag{3.12}$$

By expression (3.9), equation (3.11) leads to equation (3.4). Similarly, by expression (3.10), equation (3.12) leads to equation (3.5). This completes the proof of theorem 1.

For a reflectionless system, we consider $b(k,t)=0$. In this case, our results are reduced to those obtained by Kay and Moses (1956a). The exact solution is known as the N-soliton solution (Hirota 1971, Wadati and Toda 1972). It is found that

$$G_n(x)=\frac{c_n^2(t)}{\Delta}\,e^{\kappa_n x}\left(1+\sum_{r=1}^{N-1}\sum_{\substack{N-1C_r\\ i\neq n}}\eta^2(i_1\ldots i_r)\prod_{i=i_1}^{i_r}\eta_{ni}E_i\right). \tag{3.13}$$

Here, the symbols are defined as

$$E_n=\frac{c_n^2(t)}{2\kappa_n}\exp(2\kappa_n x) \tag{3.14a}$$

$$\eta_{ij}=(\kappa_i-\kappa_j)/(\kappa_i+\kappa_j) \tag{3.14b}$$

$$\eta(i_1\ldots i_r)=\prod^{(r)}\eta_{i_\alpha i_\beta} \tag{3.14c}$$

$$\Delta=1+\sum_{r=1}^{N}\sum_{NC_r}\eta^2(i_1\ldots i_r)\prod_{i=i_1}^{i_r}E_i \tag{3.14d}$$

where ${}_NC_r$ indicates summation over all possible combinations of r elements (designated as $i_1, i_2, \ldots, i_r$) taken from N, and (r) indicates the product of all possible pairs out of r elements. It is understood that η is unity for $r=1$.

4. Interaction of waves

Theorem 2. The solution for the inverse scattering problem may be expressed as

$$V(x,t)=\frac{2}{\pi\mathrm{i}}\int_{-\infty}^{\infty}k\frac{H^2(k,x;t)}{b(k,t)}\,\mathrm{d}k+4\sum_{n=1}^{N}\kappa_n\left(\frac{G_n(x;t)}{c_n(t)}\right)^2. \tag{4.1}$$

Proof. If we multiply equation (3.7) by $H_x-\mathrm{i}kH$, and integrate with respect to k, we get one equation. If we differentiate equation (3.7) with respect to x, multiply the resulting equation by H, and then integrate with respect to k, we get another equation. Due to symmetry of the double integrals in these two equations, by subtraction we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{-\infty}^{\infty}H\,e^{-\mathrm{i}kx}\,\mathrm{d}k$$
$$=\frac{2}{\mathrm{i}}\int_{-\infty}^{\infty}\frac{kH^2}{b(k,t)}\mathrm{d}k+\int_{-\infty}^{\infty}\mathrm{d}k\sum_{n=1}^{N}\frac{G_nH_x-H(G_n)_x-(\kappa_n+\mathrm{i}k)G_nH}{\kappa_n-\mathrm{i}k}e^{(\kappa_n-\mathrm{i}k)x}. \tag{4.2}$$

If we multiply equation (3.8) by $(G_n)_x+\kappa_nG_n$, and sum over n, we get one equation. If we differentiate equation (3.8) with respect to x, multiply the resulting equation by G_n, and then sum over n, we get another equation. Since the terms with double summation in these two equations are symmetrical in m and n, by subtraction we obtain

$$\sum_{n=1}^{N}\frac{\mathrm{d}}{\mathrm{d}x}(G_n\,e^{\kappa_n x})$$
$$=\sum_{n=1}^{N}2\kappa_n\left(\frac{G_n}{c_n(t)}\right)^2+\sum_{n=1}^{N}\int_{-\infty}^{\infty}\mathrm{d}k\frac{H(G_n)_x-G_nH_x+(\kappa_n+\mathrm{i}k)G_nH}{2\pi(\kappa_n-\mathrm{i}k)}e^{(\kappa_n-\mathrm{i}k)x}. \tag{4.3}$$

Then, substituting equations (4.2) and (4.3) into expressions (3.3) yields expression (4.1). This completes the proof of theorem 2.

Expression (4.1) represents the linear decomposition of $V(x, t)$; the integral corresponds to a continuous spectrum while the sum corresponds to a discrete spectrum. However, due to nonlinear interaction, the eigenfunctions $H(k, x)$ and $G_n(x)$ are mutually dependent as indicated by equations (3.7) and (3.8). For a general problem with an arbitrary initial condition, neither $b(k, t)$ nor $c_n(t)$ should be considered as zero.

In an attempt to solve the system of equations (3.7) and (3.8), we express equations (3.8) in the matrix form

$$(\mathbf{S}+\mathbf{I})\mathbf{\Gamma}=\mathbf{Q}, \tag{4.4}$$

where $\mathbf{S}$ denotes the square matrix, and $\mathbf{\Gamma}$ and $\mathbf{Q}$ denote the column matrices, with elements defined as

$$\Gamma_n = G_n(x)\,e^{\kappa_n x},$$

$$S_{mn} = \frac{2\kappa_m}{\kappa_m+\kappa_n}E_m,$$

$$Q_n = 2\kappa_n E_n\left(1-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{H(k,x)}{\kappa_n - \mathrm{i}k}\,e^{-ikx}\,dk\right).$$

It can be shown that elements of the inverse matrix $(S+\mathbf{I})^{-1}$ may be expressed as

$$\alpha_{mn} = -\frac{\mathbf{S}_{mn}}{\Delta}\left(1+\sum_{r=1}^{N-2}\sum_{\substack{N-2C_r\\ i\neq m,n}}\eta^2(i_1\ldots i_r)\prod_{i=i_1}^{i_r}\eta_{mi}\eta_{ni}E_i\right) \tag{4.5a}$$

for $m\neq n$, while the diagonal elements are

$$\alpha_{nn} = \left(1+\sum_{r=1}^{N-1}\sum_{\substack{N-1C_r\\ i\neq n}}\eta^2(i_1\ldots i_r)\prod_{i=i_1}^{i_r}E_i\right)\Delta^{-1}. \tag{4.5b}$$

Thus, by equation (4.4) we have

$$\Gamma_m(x) = (\Gamma_m(x))_{b=0} - \frac{1}{\pi}\sum_{n=1}^{N}\alpha_{mn}\kappa_n E_n\int_{-\infty}^{\infty}\frac{H(k,x)}{\kappa_n-\mathrm{i}k}\,e^{-\mathrm{i}kx}\,dk. \tag{4.6}$$

Substituting expression (4.6) into equation (3.7), we eventually obtain an integral equation for $H(k, x)$.

$$\frac{H(k,x)}{(b(k,t))^{1/2}}+\frac{1}{2\pi}\int_{-\infty}^{\infty}\Lambda(k,k')\frac{H(k',x)}{(b(k',t))^{1/2}}\,dk' = f(k), \tag{4.7}$$

where the kernel, expressed as

$$\Lambda(k,k') = (b(k,t)b(k',t))^{1/2}\,e^{-\mathrm{i}(k+k')x}\left(\lim_{\epsilon\to 0}\frac{1}{\epsilon-\mathrm{i}(k+k')} - \sum_{m=1}^{N}\sum_{n=1}^{N}\frac{2\alpha_{mn}\kappa_n E_n}{(\kappa_m-\mathrm{i}k)(\kappa_n-\mathrm{i}k')}\right), \tag{4.8}$$

is symmetric in the sense that k and k' are interchangeable, and the absolute term is expressed as

$$f(k)=(b(k,t))^{1/2}\,\mathrm{e}^{-\mathrm{i}kx}\left(1-\sum_{n=1}^{N}\frac{\exp(\kappa_n x)}{\kappa_n-\mathrm{i}k}(G_n(x))_{b=0}\right). \tag{4.9}$$

Now we have reduced the system of $N+1$ equations to a single integral equation. This equation shows merit over the Gel'fand–Levitan equation because the space and time variables appear only parametrically throughout. Once $H(k,x)$ is solved, the eigenfunctions $G_n(x)$ can readily be evaluated from expression (4.6). Our interest is to ascertain the class of reflection coefficients for solutions of $H(k,x)$ to exist. The reflection coefficient $b(k,t)$, and thus the eigenfunction $H(k,x)$, are assumed to possess the following basic properties:

(*a*) they are Hölder continuous;
(*b*) they have N simple poles on the imaginary axis;
(*c*) $b(-k)=b^*(k)$ and $H(-k)=H^*(k)$ by analytic continuation;
(*d*) $|b(k)|\leq 1$;
(*e*) they vanish at infinity more rapidly than $|k|^{-1}$.

Properties (*d*) and (*e*) imply the finite condition

$$\int_{-\infty}^{\infty}|b(k,t)|\,\mathrm{d}k.$$

Theorem 3. If the integral

$$\int_{0}^{\infty}|b(k,t)|\,\mathrm{d}k$$

is bounded, then the absolute term of the integral equation (4.7), as given by expression (4.9), is quadratically summa e.

Proof. By expressions (3.13) and (3.14), it can be shown that

$$\left|1-\sum_{n=1}^{N}\frac{\exp(\kappa_n x)}{\kappa_n-\mathrm{i}k}(G_n(x))_{b=0}\right|<K,$$

where K is a finite number, usually of the order of unity. Thus, we have

$$\int_{-\infty}^{\infty}|f(k)|^2\,\mathrm{d}k<K^2\int_{-\infty}^{\infty}|b(k,t)|\,\mathrm{d}k.$$

By property (*c*), this completes the proof of theorem 3 (Mikhlin 1960).

In the absence of soliton, by property (*c*), the integral equation (4.7) becomes simply

$$\frac{H(k,x)}{(b(k,t))^{1/2}}+\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\frac{(b(k,t)b(k',t))^{1/2}}{k'-k-\mathrm{i}\epsilon}\,\mathrm{e}^{\mathrm{i}(k'-k)x}\frac{H^*(k',x)}{(b^*(k',t))^{1/2}}\,\mathrm{d}k'=(b(k,t))^{1/2}\,\mathrm{e}^{-\mathrm{i}kx}. \tag{4.10}$$

In comparison with equation (4.10), the additional terms in expressions (4.8) and (4.9) apparently represent the effect of interaction due to solitons. To shed light on some physical insight, one may investigate the reflectional system with only one soliton. In this case, the integral equation (4.7) may be expressed in the form

$$\frac{H(k,x)}{(b(k,t))^{1/2}}+\frac{1}{2\pi \mathrm{i}}\int_{-\infty}^{\infty}\tilde{\Lambda}(k,k')\frac{H^*(k',x)}{(b^*(k',t))^{1/2}}\,\mathrm{d}k'=(b(k,t))^{1/2}\,\mathrm{e}^{-\mathrm{i}kx}\left(1-\frac{2\kappa}{\kappa-\mathrm{i}k}\frac{E}{1+E}\right), \tag{4.11}$$

where the kernel is given as

$$\tilde{\Lambda}(k,k')=(b(k,t)b^*(k',t))^{1/2}\,\mathrm{e}^{\mathrm{i}(k'-k)x}\left(\lim_{\epsilon\to 0}\frac{1}{k'-k-\mathrm{i}\epsilon}-\frac{2\mathrm{i}\kappa}{(k+\mathrm{i}\kappa)(k'-\mathrm{i}\kappa)}\frac{E}{1+E}\right). \tag{4.12}$$

Note that the kernel has the property

$$\tilde{\Lambda}(k,k')=\tilde{\Lambda}^*(k',k).$$

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